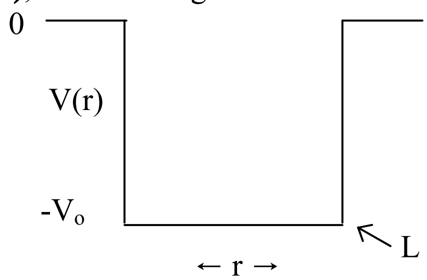
# Section 9: Forces, Potentials, and the Shell Model

As discussed in Section 7, the nucleus resembles a uniform density sphere. The forces acting on such a system can be described by a one-dimensional **square-well potential (SW)**, shown in Fig. 9.1.




$V(r) = -V_0, \quad r \le L$
$V(r) = 0, \quad r > L$

Fig. 9.1 One-dimensional square-well potential V(r) of length L.

If one solves Shroedinger's equation, $H\psi = E\psi$, for this potential

$$\frac{d^2}{dx^2}\psi - V\psi = E\psi , \text{ and}$$

$$E_n = \frac{n^2 h^2}{8mL^2} .$$

**The behavior of the wave function ψ at the walls (boundary conditions) results in quantized energy levels (eigenstates) $E_n$. Notice the dependence of the energy levels on the size of the box L, and on the principal quantum number n.**

An alternative approximation to the nuclear potential that takes into account the diffuse nuclear surface is the one-dimensional **Harmonic Oscillator (HO**) for which the force is given by Hooke's Law,

$$F = -k(x - x_0) ,$$

where x is the displacement from equilibrium and k is the force constant. If $x = x_0$, the system is at an equilibrium point because there is no force. However, if $x$ is different from $x_0$ there is a force which acts to restore the position to the equilibrium value. (Notice the negative sign.)

Integrating the force F= -dV/dx , one obtains the potential

$$V = \frac{1}{2}k(x - x_0)^2 .$$

The solution to the Schroedinger equation for this potential (Fig. 9.2) has eigenvalues

$$E_n = (n + \frac{1}{2})\hbar\omega \quad \text{where } \omega = \sqrt{\frac{k}{m}}$$

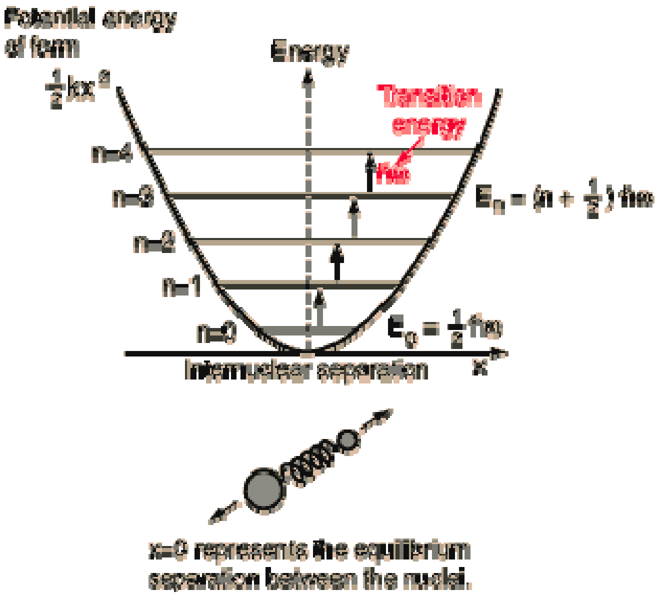


Fig. 9.2 Harmonic oscillator potential and corresponding energy levels.

Notice the energy spacing for the harmonic oscillator. What is the minimum energy of the harmonic oscillator?

Evidence for Nuclear Shell Structure

In Section 8 it was shown that the Liquid Drop model fails to describe the microscopic properties of nuclei fully. Among these features of the theory that are lacking are:

- The existence of **discrete energy levels**, indicating the need for a quantal description of the nucleus.
- Nuclei have spin I that depends on the even-odd character of the nucleus, summarized as follows:

  e-e : $I = 0$ ALWAYS

  e-o, o-e: $I = \frac{n\hbar}{2}$, where n is an odd integer (1/2, 3/2, ...)

  o-o : $I = n\hbar$, where n is an integer (0, 1, 2 ...)

  (For convenience it will be assumed that $\hbar$ is unity in the following discussion.)

  Since the spin of both the neutron and the proton is I = ½, the implication of the above result is that the pairing of nucleons of the same type is energetically favored; i.e. the nuclear force is attractive and this attraction is maximized when two nucleons are in the same orbit. In order to satisfy the Pauli Exclusion principle, the spin quantum numbers s of the pair must be opposite; i.e. one nucleon must be s = +1/2 and the other s =-1/2, for a net spin of the pair of I = 0.

- Nuclei exhibit regions of unusual stability, or closed shells, as illustrated in Fig. 8.2 of the previous section. The magic numbers occur at nucleon numbers 2, 8, 20, 28, 50, 82 and 126 (neutrons). In addition to the deviations from the liquid drop model predictions, extensive additional evidence for the closed shells exists. For example structure at the magic numbers is observed for proton, neutron and alpha particle binding energies. Lifetimes are also affected, as reflected in the half-lives of several isotopes near Z = 82 and N = 126:

$$\left\{ \begin{array}{cccc} {}^{208}_{82}Pb_{126} & {}^{209}_{82}Pb_{127} & {}^{210}_{84}Po_{126} & {}^{212}_{84}Po_{128} \\ \text{STABLE} & 22y & 138d & 10^{-7}s \end{array} \right\}$$

- Nuclei have **magnetic moments μ,** as expected for a charged particle moving in an orbit, a property that forms the basis for nuclear magnetic resonance studies. ( renamed MRI, magnetic resonance imaging by the medical profession in order to avoid the nuclear hypochondria associated with the word 'nuclear').

$$\mu = \frac{e\hbar}{2Mc} \; f(I) \, . \qquad \text{(Eq. 9.1)}$$

When the mass of the electron $M_e$ is substituted for M in Eq. 9.1, $\mu_e$ agrees with experiment to a high degree of accuracy. For protons, $M = M_p$, the mass of the proton, defines the **nuclear magneton $\mu_N$**. If the proton and neutron are elementary particles (i.e. nosubstructure), then one expects

$\mu_p = \mu_N$ and $\mu_n = 0$ , since the neutron has no charge).

Instead one observes

$$\boldsymbol{\mu_p = 2.793\ \mu_N \text{ and } \mu_n = -1.913\ \mu_N} \, .$$

Observation of the anomalous magnetic moments of the proton and neutron were one of the first indications that the proton and neutron were not elementary particles, but had substructure. Current understanding is that nucleons are composed of three quarks of two types, the **up quark** with charge **+2/3** and the **down quark** with charge **-1/3**. In this schematic model the proton is composed of two up quarks and one down quark (charge = +2/3 +2/3 – 1/3 = +1) and the neutron has two down quarks and one up quark (charge = -1/3 -1/3 + 2/3 = 0). Since some charge must exist on the periphery of the nucleon in this picture, the anomalous magnetic moments follow.

Consistent with this model, one finds the following experimental results, which supports the pairing arguments made previously:

e-e : $\mu = 0$ ALWAYS

o-e : $\mu \approx \mu_p$

e-o : $\mu \approx \mu_n$

o-o : $\mu \approx \mu_p + \mu_n$

<u>The Shell Model</u>

In the following we describe two versions of the quantum mechanical solution for nuclear structure based on the square well and harmonic oscillator potentials. The approach is analogous the that for the hydrogen atom model for periodic behavior in chemistry. The principal difference is that for atoms the electron-electron force is repulsive, whereas for nucleons the force is attractive.

From the Schroedinger equation $H\psi = E\psi$, we know that

**H = the Hamiltonian**,

which is a mathematical operator that summarizes the forces acting on the particles in the system. It is composed of two energy terms, kinetic T and potential V(r)

H = T + V(r) = kinetic + potential energy (for our purposes the SW and HO potentials).

The properties of the particles in the system are described by the **wave function** $\Psi$, which defines the probability distribution of the particles in space and time.

$$\Psi_i = f(x, y, z, t, \ldots)$$

Two important properties of the wave function are :

(1) it must satisfy the **Pauli Exclusion Principle**, which states that for Fermions (particles with half-integer spin), each quantum state must have a different wave function ; i.e. different quantum numbers

$$\Psi_i \neq \Psi_j \text{, and}$$

(2) the wave function can be described by its **parity** $\pi$, a mathematical operator that reverses the spatial coordinates of the particle

$$\pi\, \Psi(x) = \Psi(-x) = \pm \Psi$$

Mathematical functions for which $\pi\Psi(x) = +\Psi$ are said to have **even parity**; for example

$$\Psi(x) = x^2, x^4, x^6, \cos x, \text{ and s, d, g, etc. orbitals}$$

Functions for which $\pi\Psi(x) = -\Psi$ are said to have **odd parity**, for example

$$\Psi(x) = x, x^3, x^5, \sin x, \text{and p, f, h, etc. orbitals.}$$

The product of the forces H acting on the particles yields a set of discrete quantum states with energy E, each defined by a unique wave function $\Psi$. When more than one wave function is involved (as in a multiparticle system), then the total parity is the product of all the parities

$$\pi_{total} = \pi_1\, \pi_2\, \pi_3 \ldots\ldots = \pm$$

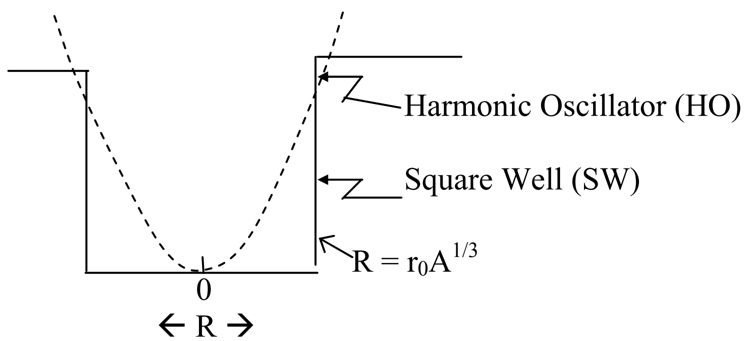
For example, the product of a sine and cosine has negative parity.

Due to the repulsive nature of the electron-electron interaction due to the Coulomb and the attractive nuclear force experienced by nucleons, one expects the models for atomic structure and nuclear structure to differ in many respects, as indicated by the following qualitative factors that apply to orbitals of the same energy.

**Atoms**, should exhibit **weak pairing** since the electrons can remain farthest apart when they are in different orbitals (Hund's Rule). **Diffuse orbitals** (i.e. low orbital angular momentum states such as $\ell = 0$, s orbitals) are preferred, again to minimize the charge density of the atomic electron cloud. And finally, the interaction between the electron spin and its orbital angular momentum **(spin-orbit coupling) should be weak** because of the size of the electron relative to that of the orbit. These assumptions are the essential modifications of the Bohr atom picture that results in the **periodic table**.

In contrast, for **nuclei**, **strong pairing** should be observed, since if both nucleons are in the same orbital, the attractive nuclear force is maximized. For this reason more **compact orbitals** should also be favored (i.e. $\ell$ = 2 d orbitals are preferred relative to $\ell$=0 orbitals. In addition, the size of the nucleon and that of the nucleus are similar in magnitude. Therefore a **strong spin-orbit** intereaction between the spin and orbital angular momenta should be expected. These arguments will come into play in the following description of the nuclear shell model.

As a starting point, let's consider the two potential energy models of Figs. 9.1 and 9.2 as applied to nuclei, both of which have mathematical solutions.




For the Square Well (uniform density sphere) the potential is:

$$V(r) = -V_0, \quad r \le R \qquad \text{(Eq. 9.2)}$$
$$V(r) = 0, \quad r > R .$$

The parabolic Harmonic Oscillator potential is given by:

$$V(r) = -V_0\,[1 - r^2/R^2] . \qquad \text{(Eq. 9.3)}$$

Of the two solutions, the HO proves to be most useful and will be elaborated here. The energy levels predicted by the HO solution are

$$E_{\nu\ell} = [2(\nu-1) + \ell]\,\hbar\omega, \qquad \text{(Eq. 9.4)}$$

where the principal quantum number $\nu$ and orbital angular quantum number $\ell$ are defined as follows:

**$\nu$ = 1, 2, 3 ... and $\ell$ = 0, 1, 2,3 ,4 ... (s, p, d, f, g ...).**

Unlike the case with atoms, the maximum value of $\ell$ is independent of $\nu$.

This quantum number notation is similar to that for the Bohr atom, although instead of the behavior shown in Eq. 9.4, the energy levels in the atomic case depend on $1/n^2$ (where n for atoms is the analogue of $\nu$ for nuclei). As in the case for atoms, each orbital angular momentum value has $(2\ell + 1)$ **magnetic substates**, $\mathbf{m_\ell = \pm \ell, \pm (\ell-1) \dots 0}$, and **intrinsic spin s = ± 1/2** values of the nucleon. As presently defined, all $m_\ell$ and s states for a given $\ell$ value have the same energy. For a given energy state $E_{\nu\ell}$, the orbital notation is written

$$\nu\ell \ .$$

Energies for the $2(2\ell + 1)$ magnetic and spin substates are the same for a given $\nu$ and $\ell$.

**Example:** What is the notation for a an HO orbital with $\nu = 2$, $\ell = 4$ ? How many nucleons can occupy this energy state?

Orbital notation: **2g**;

There are 2 spin states and $2\ell + 1$ magnetic substates or $2[(2)4 + 1]$ = **18 substates**

According to the Pauli Exclusion Principle, no two fermions can have the same set of four quantum numbers so that the $2(2\ell + 1)$ rule determines the maximum number nucleons that a given $\nu\ell$ orbital can accommodate. This number should then agree with the magic numbers observed experimentally.

Strong Spin-Orbit Coupling

In Fig. 9.3 the energy levels and their occupancy for the one-dimensional Harmonic Oscillator and Square-Well potentials are shown. From examination of the two models, as well as the amalgamation of the two shown in the center of the plot, it is apparent that none of the models can successfully reproduce the magic numbers at 2, 8, 20, 28, 50, 82 and 126. One encounters similar problems with the Bohr model in trying to explain the magic numbers of 2, 8, 18, 36, 54 and 86 in the periodic table. In both cases additional physics assumptions are required.

The solution to the mismatch between experiment and the simple nuclear shell model was proposed by Maria Goeppert Mayer and Hans Jensen, for which they received the Nobel Prize in 1963. They introduced an empirical correction to the Harmonic Oscillatror model that argued that as follows. As a consequence of the strong nuclear force and the compactness of nuclear orbital, there should be a **strong coupling between the particle spin s and its orbital angular momentum** $\ell$. This assumption requires the introduction of a new quantum number j, defined by the total angular momentum of the particle:

$$j = \vec{\ell} + \vec{s} = \ell \pm ½$$

In this new framework the orbital notation becomes

$$\nu\ell_j \ .$$

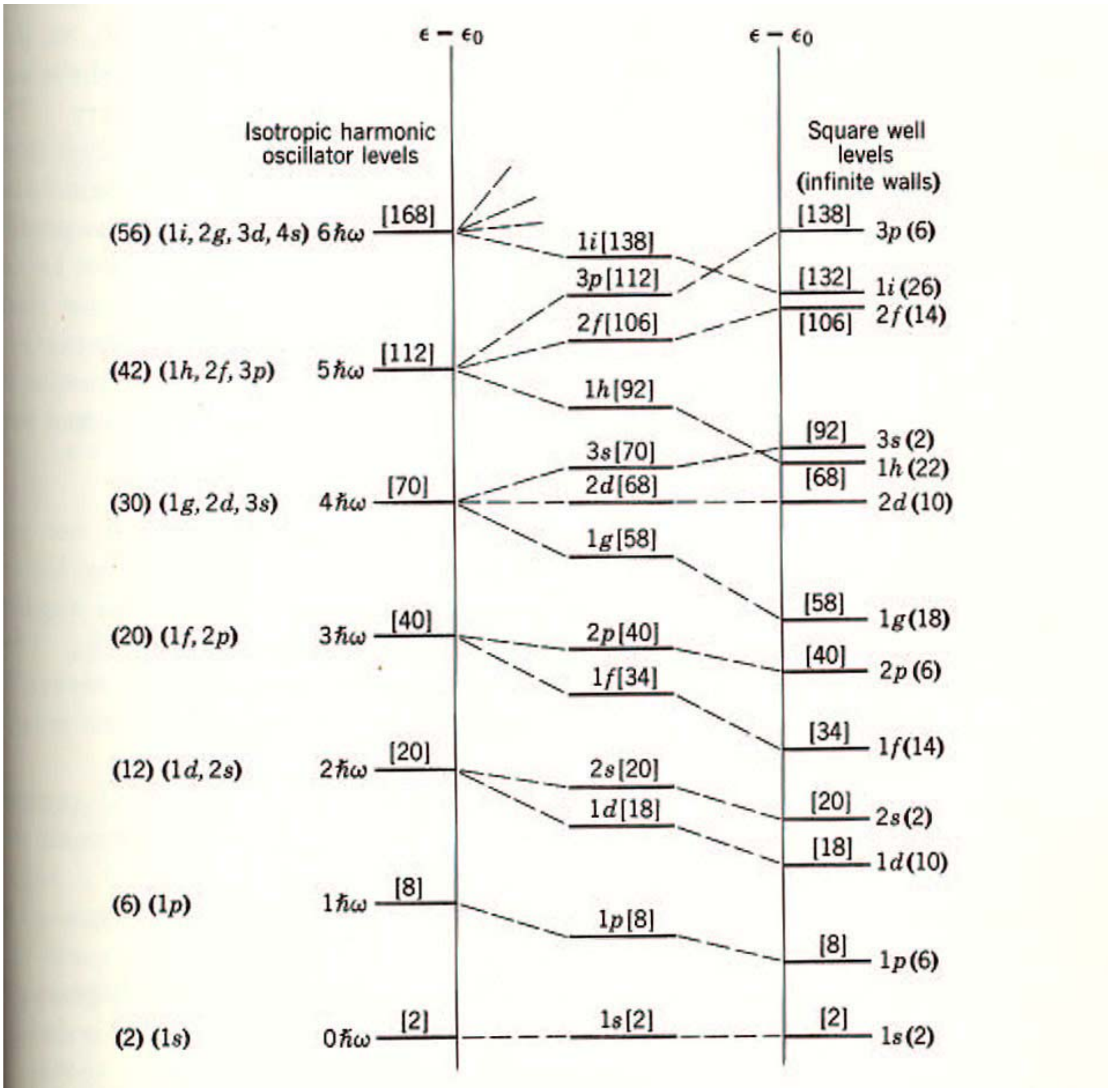


Fig. 9.3 Predicted energy levels based on the HO (left) and SW (right) potential models. The maximum number of nucleons that a given level can hold is shown in parentheses is shown in parentheses next to the spectroscopic notation. The sum of nucleons in all energy levels up to a given level is shown in brackets. The states listed in the center of the plot show an amalgamation of the two models

Under the strong spin-orbit assumption, the four quantum numbers become

$\nu$ (remains the same)

$\ell$ (remains the same)

$j = \ell \pm ½$, and

$m_j = +j, (j-1) \ldots -j$ for which there are $2j + 1$ values

**Example**: What is the spectroscopic notation for an orbital with $\nu = 1$ and $\ell = 2$? What are the magnetic substates?

$\ell = 2$ is a d state

$j = 2 \pm 1/2 = 3/2, 5/2$
Therefore the spectroscopic notation is $1\,d_{3/2}$ & $1\,d_{5/2}$
The magnetic substates are:
**$j = 3/2$**, $m_j = 3/2, 1/2, -1/2, -3/2 = 2j + 1 =$ **4 possible values**
**$j = 5/2$**, $m_j = 5/2, 3/2, 1/2, -1/2, -3/2, -5/2 = 2J + 1 =$ **6 possible values**
Net result: we still have ten d states but now they are split into two energy levels.

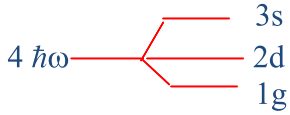
The splitting of the angular momentum states into two j-states leads to an alteration of the energy level spectrum. The order and degree of splitting are governed by three rules:

(1)For the **same oscillator energy** as defined in Eq. 9.4, states with the highest angular momentum lie lowest in energy; i.e.

$$\mathbf{E_{\ell} < E_{\ell-2} < E_{\ell-4} \ldots}$$

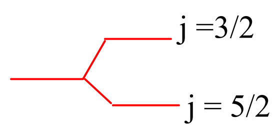
In other words, more compact orbitals (higher $\ell$) permit stronger nuclear attraction. This is just the opposite of the angular momentum correction for atoms in the Bohr model that is necessary to account for the periodic table.

**Example**: What is the order of j substate energy levels for the HO energy $E^{HO} = 4\hbar w$?
From Eq. 9.4, three possible states can yield $E^{HO} = 4\hbar w$: 1g, 2d and 3s.
The rule says that the energy splitting is therefore




(2) For the same $\nu\ell$, coherent coupling between the spin and orbital angular momentum, $j = \ell + ½$, lies lower in energy than states in which the spin and orbital angular momentum are anti-correlated, $j = \ell - ½$.

$\mathbf{E_{\ell+1/2} < E_{\ell-1/2}}$ ; e.g. $\ell = 2$ (d state)




(3) The larger the orbital angular momentum, the larger the energy splitting $\Delta E_j$ between j states.

$$\mathbf{\Delta E_j \propto \ell}$$

These assumptions lead to a rearranged level order for the HO, as shown in fig. 9.4.

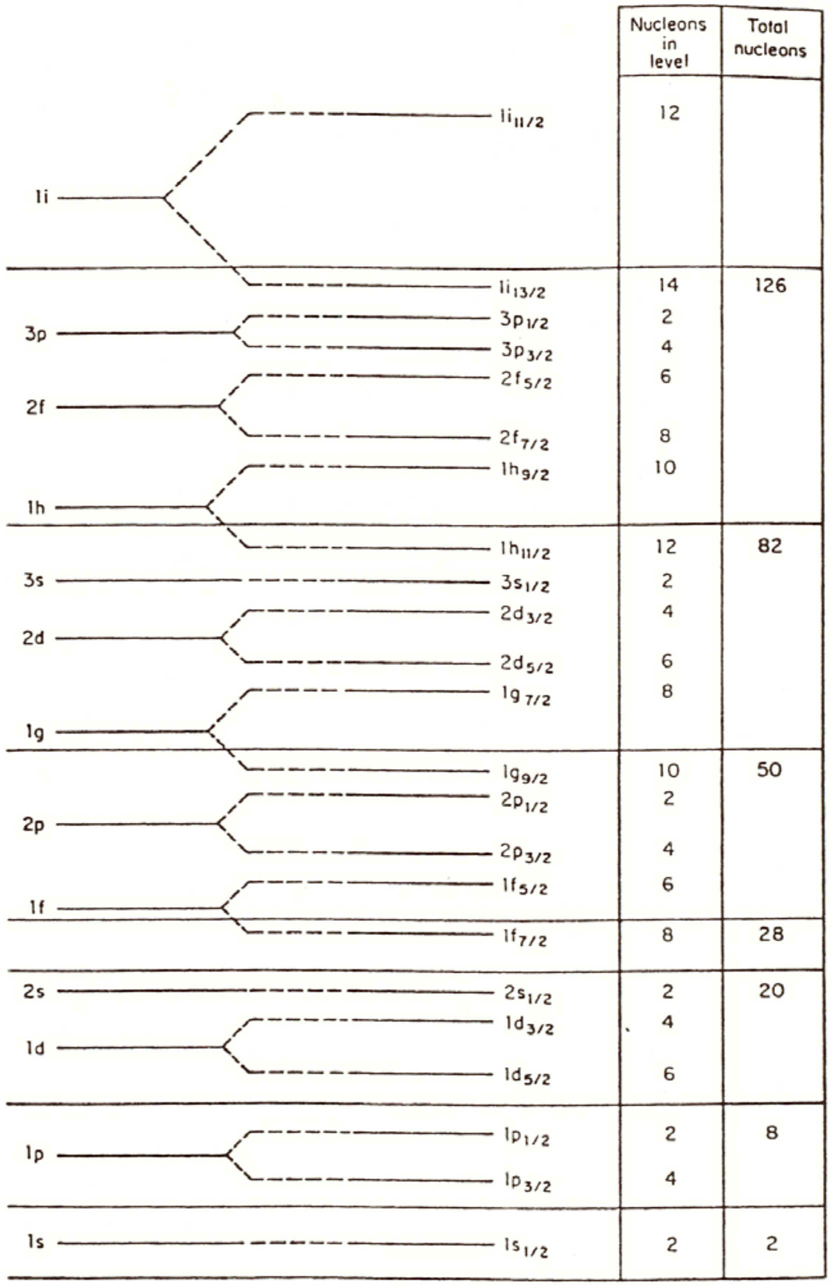



Fig 9.4 Energy levels in the shell model with strong spin-orbit coupling. Levels at the far left are for the basic HO potential (Fig.9.3 and Eq. 9.4). Levels and their splitting are shown in the middle. The first column gives the number of nucleons in each j state and the second column give the cumulative number of nucleons.

The splitting of $\ell$ states into two j states and the increasing gap between the two j states causes the highest angular momentum state for a given $\ell$ to be pushed down into the level below it. For example, as shown in fig.9.4, the ten nucleons in the $1g_{9/2}$ level are now similar in energy to the 2p and 1f orbitals, creating an energy gap (shell) between the other 3s, 2d and 1g levels. As a consequence, the shell closure occurs at 50 nucleons instead of 40, bringing the model into agreement with observation. With the modifications introduced to the HO potential model, Fig. 9.4 shows that the nuclear closed shells at 2, 8, 20, 28, 50, 82 and 126 can now be described with logical physical assumptions. The filling of nuclear levels in the shell model parallels that of filling electrons in the Bohr atom model; i.e. the lowest energy levels are filled sequentially up to the number of particles available.

Prediction of Nuclear Spins and Parities

Fig. 9.4 serves as a first-order guide to the prediction of nuclear spins and parities. The ground rules for interpreting the energy level diagram depend on the even-odd character of the nucleus.

- **Even-Z – Even-N Nuclei**
  In even-even nuclei all proton and neutron levels are filled pairwise and therefore their spins cancel out: i.e. each $\{\nu, \ell, j, m_j\}$quantum state is matched with a $\{\nu, \ell, j, -m_j\}$ state. Therefore the spin I of the nucleus is 0 and its parity $\pi$ is positive ($+\bullet+ = +$ and $-\bullet - = +$). Therefore, the spin and parity of all even-even nuclei is

  $$I\,\pi = 0\,+$$.

  There are **no exceptions** to this rule. Thus we can safely predict that the unknown nucleus $^{310}126$ has spin I = 0 and parity $\pi = +$.

- **Odd-A Nuclei**
  For odd-A nuclei the **independent particle assumption** is invoked to predict the total spin and parity of a nucleus. This simple model assumes that the nucleus is composed of an even-even core and a single odd nucleon. The nucleons in the even-even core fill all the lowest energy levels and the odd nucleon occupies the level corresponding to the highest unfilled level. Therefore the **core** has a spin and parity of **$I\,\pi = 0\,+$** and the **spin and parity of the odd particle**, as determined from the shell model, **define the spin and parity of the entire nucleus**

  $$I = \vec{I}\text{ (core)} + \vec{I}\text{ (last nucleon)} \quad = \quad 0 + j = j$$
  $$\pi = \pi\text{ (core)} \times \pi\text{ (last nucleon} \quad = \quad + \bullet \pm = \pm$$

  **Example**: Predict the spin and parity of the following nuclei: $^{119}$In and $^{47}$Ca.
  (1) $^{119}$In can be broken down into its core plus odd proton as follows:

  $$^{119}_{49}\text{In} = {}^{118}_{48}\text{Cd} + p$$

The odd particle is the 49th proton. Referring to Fig. 9.4, the first 40 particles of the core fill the levels up to the $1g_{9/2}$ state and the last eight occupy this level. Since the $1g_{9/2}$ orbital can accommodate ten particles, the 49th proton also goes in this orbital also. Thus the spin is 9/2 and the parity is positive because g orbitals ($\ell = 4$) have even parity.

Shell model prediction: $I\pi = 9/2+$ , which is observed.

(2) Similarly $^{47}Ca$ can be written

$$ {}^{47}_{20}\mathrm{Ca} = {}^{46}_{20}\mathrm{Ca} + {}^{1}_{0}\mathrm{n}. $$

In this case the odd neutron is the 27th particle. Counting up from the bottom in Fig. 9.4, the 27th particle will occupy the $1f_{7/2}$ level. This predicts the the spin will be 7/2 and the parity negative since $\ell = 3$ for f orbitals.

Shell model prediction: $I\pi = 7/2^-$ , which is observed.

As a rule the shell model predictions for odd-A nuclei works well for nuclei near the closed shells, where nuclei are spherical in shape. Between the closed shells, deviations are observed because nuclei become deformed into prolate spheroids and the first-order model presented here does not include a shape degree of freedom.

- **Odd-Odd Nuclei**

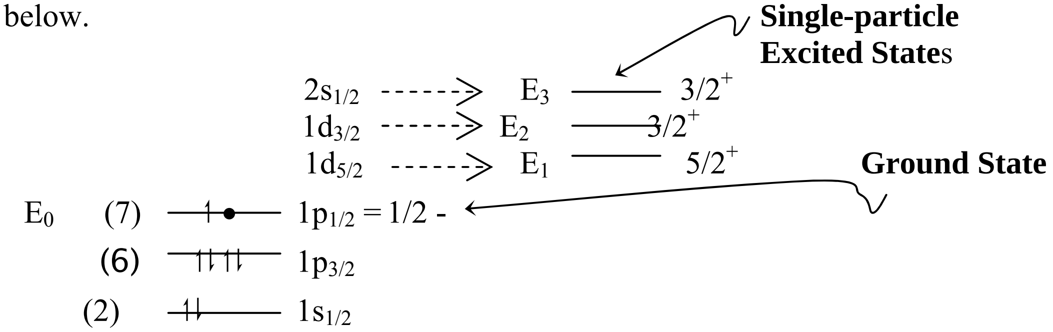
The case of odd-odd nuclei introduces the need to couple the orbital angular momentum vectors of the last odd proton to last odd neutron

$$ I = \vec{j}_n + \vec{j}_p = (j_n + j_p), \ldots (j_n - j_p). $$

This angular momentum coupling leads to a series of possible spins. Consequently predictions for odd-odd nuclei are difficult to predict, although a set of vector addition rules, the Brennan-Bernstein Rules, provide reasonable estimates, but will not be covered in this discussion.

Excited States

The shell model can also be used to predict the spins and parities of particle states in excited nuclei. In this approach the last single particle is promoted to successively higher energy states as the nucleus is excited. Consider the excited states of the isotope $^{15}O$ shown below.



$^{15}O$ has an even number of protons, so its contribution to the spin and parity is $I\pi = 0+$. The seven neutrons fill the $1s_{1/2}$, $1p_{3/2}$ and $1p_{1/2}$ levels sequentially, so that the odd neutron falls in the $1p_{1/2}$ level, resulting in a ground-state spin and parity of $I\pi = ½-$. When the nucleus is excited, the odd neutron can be promoted to the unoccupied higher energy levels $1d_{5/2}$, $1d_{3/2}$ and $2s_{1/2}$, for which the spins and parities are $5/2^+$, $3/2^+$ and $3/2^+$, respectively. Thus, the shell model also finds application in predicting the spins and parities of excited states, especially near the closed shells.

The prolate deformation of nuclei between the closed shells leads to collective motion, just as in the case of diatomic molecules. The spectra for these nuclei become more complex, exhibiting rotational and vibrational bands superimposed on the single-particle levels of the shell model.

## The Shell Model and the Real World

Both the Bohr model of the atom and the nuclear shell model are first-order models. Nonetheless, they are valuable tools for systematizing the properties of atoms and nuclei. The successes of the nuclear shell model with strong spin-orbit coupling are:

- The closed shells are predicted successfully.
- Spin, parity and magnetic moment predictions are always correct for e-e nuclei and usually correct for odd-A nuclei near the closed shells where nuclei are spherical. Prediction of values for o-o nuclei is problematic.
- The spins and parities of low-lying single-particle levels in odd-A nuclei are predicted relatively well near the closed shells.

In more sophisticated approaches to nuclear structure, refinements of the physics in the model lead to improved descriptions of experimental level structure. For example, instead of a spherical HO potential, shape degrees of freedom can be introduced into the potential, e.g.

$$V(r) = V_0 \left(1 - \left[\frac{a\,x^2}{R^2} + \frac{b\,y^2}{R^2} + \frac{c\,z^2}{R^2}\right]\right),$$

which allows for shape deformations. Computation techniques that approximate the Fermi-function shape of the nuclear potential have further expanded the success of the model.